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Can quantum mechanics and supersymmetric quantum mechanics be the multidimensional Ermakov theories?*

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Abstract. For both the Schrödinger equation in quantum mechanics and the Riccati-type equation satisfied by the superpotential in supersymmetric quantum mechanics, we explicitly show that there exists an Ermakov-type functional invariant with respect to the space variable. An energy-like interpretation is suggested for this invariant.

More than a century ago, Ermakov [1] originally suggested a connection between the solutions of a pair of coupled differential equations. In recent years, Ray and Reid in a series of papers [2–6] and several other authors [7–12] have exploited such a connection in the studies of time-dependent (TD) harmonic oscillators and with different degrees of generalizations. During the course of their studies, Ray and Reid have evolved a method of constructing the invariant for one-dimensional TD systems known as the Ermakov method and, accordingly, the invariant so constructed as the Ermakov invariant. While mathematical aspects of these $(1+1)$-dimensional Ermakov systems have been studied by Athorne [13] at a somewhat deeper level, Ermakov-like systems in $(2+1)$ dimensions have also been investigated [14] recently at the classical level.

On the other hand, Korsch and his co-workers [15, 16], and subsequently Lee [17], noticed an interesting and striking *similarity* between the classical equation of motion for a TD harmonic oscillator and the Schrödinger equation for an arbitrary potential. This has further enhanced the domain of applicability of the Ermakov theory to various physical problems.

It is now well known that a TD harmonic oscillator in $(1+1)$ dimensions described by

$$\ddot{x}(t) + \omega^2(t)x(t) = 0 \tag{1}$$

admits an Ermakov invariant

$$l = \left(\frac{x}{\rho}\right)^2 + c^{-2}(\dot{x}\rho - x\dot{\rho})^2 \tag{2}$$

where the auxiliary variable, $\rho(t)$, satisfies the equation

$$\ddot{\rho}(t) + \omega^2(t)\rho(t) = c^2/\rho^3(t). \tag{3}$$

* Dedicated in honour of Professor H J W Müller-Kirsten on his 60th birthday.
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Korsch and co-workers [15, 16] and also Lee [17] have made use of the existence of similarity between equation (1) and the Schrödinger equation

$$\psi''(x) + k^2(x)\psi(x) = 0 \tag{4}$$

where $k^2(x) = 2m(E - V(x)/\hbar^2$. In fact, they merely identify $x(t)$, $\omega(t)$ and $\rho(t)$ with the wavefunction $\psi(x)$, local wavenumber $k(x)$ and $A(x)$, respectively, in an *ad hoc* manner and accordingly write the form of the Ermakov invariant as

$$I = \left(\frac{\psi}{A}\right)^2 + c^{-2}(\psi' A - \psi A') \tag{5}$$

with $A(x)$ satisfying

$$A''(x) + k^2(x)A = c^2/A^3(x). \tag{6}$$

Further, a connection between the solution of equation (4) and that of equation (6) (the latter is known as Milne's equation in the literature [15, 18]) obtained as

$$\psi(x) = NA(x)\sin\left(c\int^x A^{-2}(x)\,\mathrm{d}x - \delta\right) \tag{7}$$

leads to a new quantization rule

$$c\int_{-\infty}^{\infty} A^{-2}\,\mathrm{d}x = (n+1)\pi \qquad (n = 0, 1, 2, \ldots) \tag{8}$$

which is called the Milne quantization condition. Applications of these results to a number of physical problems are discussed by Korsch and coworkers and Lee. Equations (4)–(6) define a multidimensional Ermakov system and the invariant (5) is termed [17] as the configurational space Ermakov invariant.

The purpose of the present paper is simple; however, it is nevertheless intriguing enough to warrant a serious investigation. In fact, what we shall show here is that the *ad hoc* identification of $x(t)$, $\omega(t)$ and $\rho(t)$ with $\psi(x)$, $k(x)$ and $\rho(x)$, respectively, as has been done in the past [15–17] to establish quantum mechanics (QM) as a multidimensional Ermakov theory, was *not at all* necessary. Instead, a particular class of solutions of the Schrödinger equation (4) might demand the existence of not only an Ermakov-type invariant but also the Milne quantization condition (8) in a natural way rather than the *ad hoc* identification of the variables. Also, the Riccati-type equation satisfied [19, 20] by the superpotential in the case of supersymmetric quantum mechanics (SUSYQM) follows the Ermakov-type description in a natural way, as will be emphasized later. In spite of the fact that (i) Ermakov theory has now been known for more than 110 years, (ii) traditional (Schrödinger) QM is already close to 70 years old and (iii) SUSYQM has also been growing for the past decade, strangely enough these transparent connections between the QM or SUSYQM and Ermakov theory have remained unnoticed thus far to the best of our knowledge. As a matter of fact, such solutions in QM or SUSYQM may turn out to be much richer than the previously known ones, as far as the physical content in their structure is concerned. Even the well known WKB [21] (or for that matter supersymmetric WKB [22]), quantization rule could be studied as a special case of the Milne quantization condition, if the latter is derived independently of the corresponding classical description.

Independent of the form of $k^2(x)$ (or equivalently that of $V(x)$ appearing through $k^2(x) = 2m(E - V(x))/\hbar^2$), let us make an ansatz for the solution of the time-independent

Schrödinger equation (4) as†

$$\psi(x) = N_1 A(x) \exp(\mathrm{i}S(x)). \tag{9}$$

Use of this form in (4) yields

$$A'' + \mathrm{i}(AS'' + 2S'A') + A[k^2(x) - S'^2] = 0. \tag{10}$$

On equating the real and imaginary parts of this equation separately to zero, one finds

$$A'' + A[k^2(x) - S'^2] = 0 \tag{11a}$$

$$AS'' + 2S'A' = 0. \tag{11b}$$

Here, equation (11b) can be solved immediately to give

$$S' = c/A^2 \qquad \text{or} \qquad S = c\int A^{-2}\,\mathrm{d}x - \delta \tag{12}$$

where δ is a constant of integration. Using this form of S' in equation (11a), one arrives at the same equation as (6) for $A(x)$. Now eliminating $k^2(x)$ from equations (6) and (4) (as one does in the method of Ray and Reid [2–6]) one immediately obtains the Ermakov-type invariant

$$K = c^{-2}(\psi' A - \psi A')^2 + \left(\frac{\psi}{A}\right)^2 \tag{13}$$

which is of the same form as (5) with I replaced by K. This replacement is mainly to differentiate it from the corresponding classical case (cf equation (2)). In fact, the functional form K here is a constant with respect to the space evolution of the system, i.e. $\mathrm{d}K/\mathrm{d}x = 0$. Also note that for certain restrictions on $\psi(x)$, the ansatz (9) along with (12) gives rise to the connection (7) between $\psi(x)$ and $A(x)$ and subsequently the other connections [15] between the general solutions of (6) and (4). Further, the Milne quantization condition (8) follows as before [15, 17] by just imposing the requirement that the wavefunctions be bounded at both ends of the interval.

It is well known [19, 20] that in the supersymmetric formulation of quantum mechanics the superpotential, $W(x)$, is derived from the ground-state wavefunction Ψ_0 as

$$W(x) = -\Psi_0'/\Psi_0 \qquad \text{or} \qquad \Psi_0(x) = N_0 \exp\left(-\int^x \mathrm{d}y\, W(y)\right). \tag{14}$$

With a knowledge of Ψ_0 and the ground-state energy E_0, one can factorize the Hamiltonian in the form

$$2H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) = A^+A + 2E_0 \tag{15}$$

where $A = \mathrm{d}/\mathrm{d}x + W$ and $A^+ = -\mathrm{d}/\mathrm{d}x + W$. The pair of Hamiltonians related by the supersymmetry are

$$2H_\pm = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\epsilon \tag{16}$$

where, for convenience, we have used $\epsilon = \pm$, and V_ϵ is given by

$$V_\epsilon(x) = W^2(x) + \epsilon W'(x) \tag{17}$$

† Note that the same form of the Schrödinger equation (as equation (4)) can be retained for the central potentials in the three-dimensional case in terms of the reduced form of the Schrödinger equation and with $V(x) \to V(r) + \ell(\ell+1)\hbar^2/2mr^2$ (cf [21]).

and $H_{-}\Psi_0 = 0$. Conversely, for the given form of $V(x)$ (or of $V_\epsilon(x)$ for that matter) one can determine $W(x)$ and consequently the ground state Ψ_0 by solving the Riccati equation (17). It may be mentioned that in SUSYQM an equation exactly similar to (17) also arises [20] when one investigates the excited states of the system for the potential $V(x)$. In that case, however, there also appears a constant term in (17) which can be accounted for through a redefinition of $V_\epsilon(x)$. Therefore, the nature of the basic equation to be solved remains essentially the same, viz.,

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \epsilon V_\epsilon(x) - \epsilon W^2. \tag{18}$$

As a matter of fact, one can recover the same form of the Schrödinger equation as (4) from (18) just by using the Riccati transformation [23] $W(x) = -\Psi_0'/\Psi_0$, which happens to be the same as the defining equation (14) for the superpotential $W(x)$ in the present case. In other words, equation (18) represents an alternative form of the Schrödinger equation. Hence, whatever we have discussed earlier in connection with quantum mechanics and Ermakov theory, the same arguments remain valid here for the SUSYQM case. Equations of the type (4), (13) and (6) can easily be derived and consequently imply SUSYQM as a multidimensional Ermakov theory.

We would like to conclude with the following remarks. One of the interesting aspects of the present work is to emphasize the fact that the conventional WKB approximation [21] turns out to be a special case of the ansatz (9). In fact, by assuming the slow variation of $k(x)$ with respect to x and subsequently neglecting A'' as compared to A', from equation (11) one arrives at

$$S' = k(x) \qquad \text{or} \qquad S = \int^x k(x)\,\mathrm{d}x + \delta_1$$

and $A = A_0/\sqrt{S'}$. As a result it is not difficult to realize that the standard WKB quantization rule is obtained as a special case of the Milne quantization condition (8). The latter is found [15, 24] to have a superior numerical stability for $n \gg 1$ over the WKB results.

In view of the fact that (i) Ermakov invariants, basically, are the angular momentum-type invariants (cf equation (2) or (3)) and (ii) at the classical level I (cf equation (2)) is interpreted [12] as the angular momentum in a projected two-dimensional plane, we suggest here an energy-like interpretation for K (or for I). As far as equation (13) is concerned one can look at this form in two different ways.

(1) Define a new function $\phi = \psi/A$, which allows us to write (13) as

$$K = c^{-2}A^4(\phi')^2 + \phi^2. \tag{19}$$

By a suitable transformation, the x-dependence of the coefficient A^4 of ϕ'^2 in (19) can be transferred to the coefficient of ϕ^2, thereby implying K as an x-dependent harmonic oscillator-like Hamiltonian in the ϕ variable.

(2) In view of the several generalizations of the Ermakov systems at the classical level, particularly by Ray and Reid [4, 5], one can easily write the functional form (13) at the quantum level as

$$K = \frac{L_{\mathrm{fn}}^2}{2\mathcal{I}} + f(\psi, A) \tag{20}$$

where L_{fn} is the angular momentum-like quantity defined in function space, i.e. in the ψ–A plane, and $\mathcal{I}$ can be identified with a moment of inertia-like quantity. On the other hand, the energy of a physical system is always defined as

$$E = \frac{P^2}{2m} + V(x) \tag{21}$$

where symbols have their usual meanings. While a comparison of (20) and (21) shows a perfect analogy between K and E for each kinetic and potential terms, mathematically both K and E are the conserved quantities belonging to different spaces. Thus, K again appears as an energy-like invariant of the system but in a different space.

To summarize, we mention that both the Schrödinger equation in QM and Riccati equation satisfied by the superpotential in SUSYQM clearly have been shown to have their basis in Ermakov theory which was discovered in the last century. In fact, such connections appear more in a natural fashion in the present work than in an *ad hoc* style employed by others, by way of demanding the existence of a functional invariant with respect to the space variable. No doubt the detailed physical understanding of this invariant (at both classical and quantum level) requires further investigation; the energy-like interpretation advanced here may well be a plausible one. Finally, it is imperative to emphasize that the word 'multidimensional' in this work refers to the Hilbert space *vis-à-vis* the Schrödinger equation and it is not to be confused with the spatial degrees of freedom.

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